Journal of Mechanics Engineering and Automation 9 (2019) 64-68
doi: 10.17265/2159-5275/2019.02.004

Power Series Solution for Strongly Non-linear Conservative Oscillators

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Abstract: This paper compares the accuracy of the power series approach with that of the modified Lindstedt-Poincare method for strongly nonlinear vibration. The free vibration of an undamped Duffing oscillator is considered because it has an exact solution. In the power series approach, the time variable is transformed into an "oscillating time" which reduces the governing equation to a form well-conditioned by the power series method. The results show that the power series approach provides extremely accurate vibration frequencies, even at large values of the nonlinear parameter, compared with errors of up to nine percent for the modified Lindstedt-Poincare method.

Key words: Nonlinear , oscillation , power series , conservative.

1. Introduction

The Lindstedt-Poincare method is a well-known perturbation technique which provides uniformly valid asymptotic expansions for weakly nonlinear vibrations [1-3]. Several authors have attempted to modify the standard Lindstedt-Poincare method to extend its validity to strongly nonlinear vibrations [4-6]. In 2002, Ji-Huan He [7], proposed a modified L-P method by expanding a constant in powers of the expanding parameter to avoid secular terms in the perturbation series. The proposed method was applied to some examples and the results showed that the solutions for strongly non-linear problems were uniformly valid on the whole solution domain.

In 1996, a power series approach for the study of periodic motion [8] was proposed. This approach is based on transforming the time variable into an "oscillating time variable" which transforms the governing equation into a form well-conditioned for a solution by the power series method.

In this paper, a comparison is made between the accuracies of the modified Lindstedt-Poincare method and the power series approach for strongly nonlinear vibration. The undamped Duffing oscillator is considered for which an exact solution is available. It is worth mentioning that both techniques apply to undamped vibrations only.

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2. Duffing Equation

For the purpose of comparison, consider the free vibration of an undamped Duffing oscillator governed by the equation:

$$\ddot{u} + u + \epsilon u^3 = 0, \tag{1}$$

Subject to the initial conditions $u(0) = A$, $\dot{u}(0) = 0$. The constant ϵ is the nonlinear parameter.

This problem has an exact solution [9] expressible in the form of elliptic functions which represent periodic motions whose periods are given by elliptic integrals.

The power series technique can be used to capture periodic motions by transforming [8] the time variable into a new independent variable τ as:

$$\tau = \sin \omega t \tag{2}$$

which oscillates between the values of -1 and +1 at a frequency of ω as time t is increased indefinitely. The infinite time domain ($0 \le t \le \infty$) is thereby reduced to a finite time scale ($-1 \le \tau \le 1$). When this

transformation is carried out by Eq. (1), the transformed differential equation and initial conditions become:

$$\omega^2(1-\tau^2)u'' - \omega^2\tau u' + u + \epsilon u^3 = 0,$$
$$u(0) = A,\ u'(0) = 0. \quad (3)$$

where the prime denotes differentiation with respect to τ. The oscillating time frequency ω is to be determined.

It is now assumed that a convergent power series expansion about τ = 0 exists for:

$|\tau| \prec 1$, as

$$u(\tau) = a_1 + a_2\tau + a_3\tau^2 + \cdots = \sum_{n=1}^{\infty} a_n \tau^{n-1} \quad (4)$$

where a_i is constant coefficients to be determined. Using Eq. (4), the different terms in Eq. (3) can be written as:

$$\omega^2 u'' = \omega^2 \sum_{n=1}^{\infty}(n-1)(n-2)\, a_n \tau^{n-3} = \omega^2 \sum_{n=1}^{\infty} n(n+1)\, a_{n+2}\tau^{n-1} \quad (5)$$

$$-\omega^2\tau^2 u'' = -\omega^2 \sum_{n=1}^{\infty}(n-1)(n-2)\, a_n \tau^{n-1} \quad (6)$$

$$-\omega^2\tau u' = -\omega^2 \sum_{n=1}^{\infty}(n-1)\, a_n \tau^{n-1} \quad (7)$$

$$u^3 = \sum_{n=1}^{\infty} b_n\, \tau^{n-1} \quad (8)$$

A shift of index has been made in Eq. (5) so that all the terms involved have the same form. The constant b_i in the nonlinear term, Eq. (8), can be determined once all the coefficients $a_1, a_2, \ldots\ldots, a_i$ have been computed.

Substituting Eqs. (4)-(8) into Eq. (3) and equating coefficients of each power to zero, gives the recurrence relation:

$$a_{n+2} = \frac{[(n-1)^2\omega^2 - 1]a_n - \epsilon b_n}{n(n+1)\omega^2}, \quad n = 1, 2, \ldots \quad (9)$$

between the series coefficients. The fundamental coefficients a_1 and a_2 are determined by imposing the initial conditions giving:

$$a_1 = u(0) = A, \quad a_2 = \frac{u'(0)}{\omega} = 0 \quad (10)$$

It follows from Eqs. (9) and (10) that all the even coefficients associated with odd powers vanish and as a result, the expansion, Eq. (4) captures the periodic motion every half cycle of the oscillating time. Consequently, the oscillating time frequency ω is one half of the vibration frequency Ω.

$$\omega = \Omega/2 \quad (11)$$

Since the odd power coefficients turned out to be zero, Eq. (4) can be written in compact form as:

$$u(\tau) = a_1 + a_2\tau^2 + a_3\tau^4 + \cdots = \sum_{n=1}^{\infty} a_n\, \tau^{2n-2} \quad (12)$$

and the recurrence Eq. (9) becomes

$$a_{n+1} = \frac{[\omega^2(2n-2)^2 - 1]a_n - \epsilon b_n}{2n(2n-1)\omega^2}, \quad n = 1, 2, \ldots \quad (13)$$

The frequency ω can be computed from Rayleighs energy principle for conservative systems which equates the maximum potential and kinetic energies.

For the system considered, the kinetic and potential energies *T* and *V* are respectively given by:

$$T = \frac{1}{2}\dot{u}^2 = \frac{1}{2}\omega^2(1-\tau^2)\acute{u}^2 \quad (14)$$

$$V = \frac{1}{2}u^2 + \frac{1}{4}\epsilon u^4 \quad (15)$$

The maximum potential energy occurs at τ = 0 when the displacement is maximum $u(0) = A$, and the maximum kinetic energy occurs at the equilibrium position when τ = $\pm 1/\sqrt{2}$. By using Eqs. (12), (14), and (15), Rayliegh energy principle can be written as:

$$\frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4 = \frac{\omega^2}{2}\left(a_2 + a_3 + \frac{3a_4}{4} + \cdots\right)^2 \quad (16)$$

Eq. (16) is the characteristic equation of the system since all the series coefficients involved are functions of frequency ω, as given by Eq. (13). It is found that Eq. (16) has two roots, the first simply changes sign of the error function ($E = V_{max} - T_{max}$), and the second

root which is the correct one makes the magnitude of the error $|E|$ a minimum. Once the frequency ω is obtained, the vibration frequency is then twice that value and the corresponding series coefficients uniquely determine the periodic motion, which can be written as:

$$u(t) = a_1 + a_2 sin^2 \omega t + a_3 sin^4 \omega t + \cdots \quad (17)$$

3. Results and Discussion

Power series solutions were obtained for Eq. (1) by using the recurrsive Eq. (13) with the amplitude A = 1. Before comparing the results with the exact solution and those of the modified L-P technique, a convergence test was carried out for large nonlinearity (∈ = 50). Fig. 1 shows the convergence of the oscillating time frequency ω as the number of terms in Eq. (12) is increased.

Table 1 compares the vibration frequency obtained using different methods for various values of the nonlinear parameter ϵ. The results show that the power series method provides extremely accurate frequencies even at large nonlinearity. The error of the modified L-P technique [7] exceeds the 7% limit predicted by the author but remains within 9% of the exact values for the range of ϵ considered.

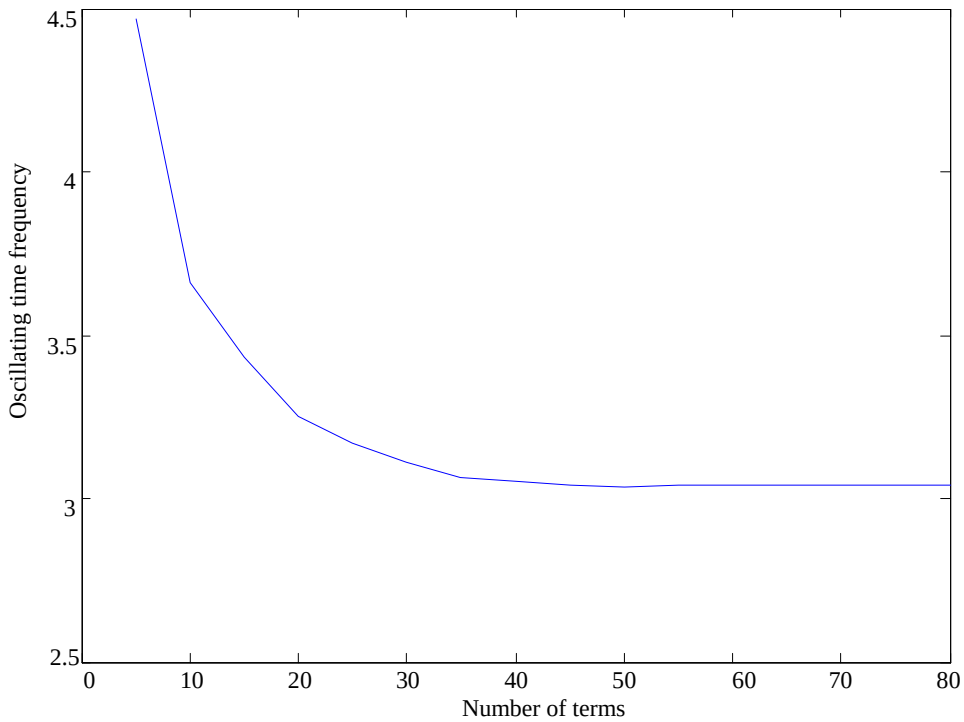

Fig. 1 Convergence of the oscillating time frequency (A = 1, ∈ = 50).

Table 1 Comparison of vibration frequency.

Nonlinear parameter ϵ	Modified L-P	Power series	Exact solution
1	1.2553	1.3178	1.3178
10	2.6253	2.8666	2.8666
20	3.5799	3.9240	3.9240
30	4.3291	4.7516	4.7516
40	4.9665	5.4548	5.4547
50	5.5309	6.0762	6.0771

The time response of the oscillator given by the power series method , was compared with a numerical solution based on a fourth order Runge-kutta algorithm for A = 1, ϵ = 2. Figs. 2 and 3 compare respectively the displacement and velocity responses. Excellent agreement is seen in each response.

The first thirty series coefficients are shown in Table 2.

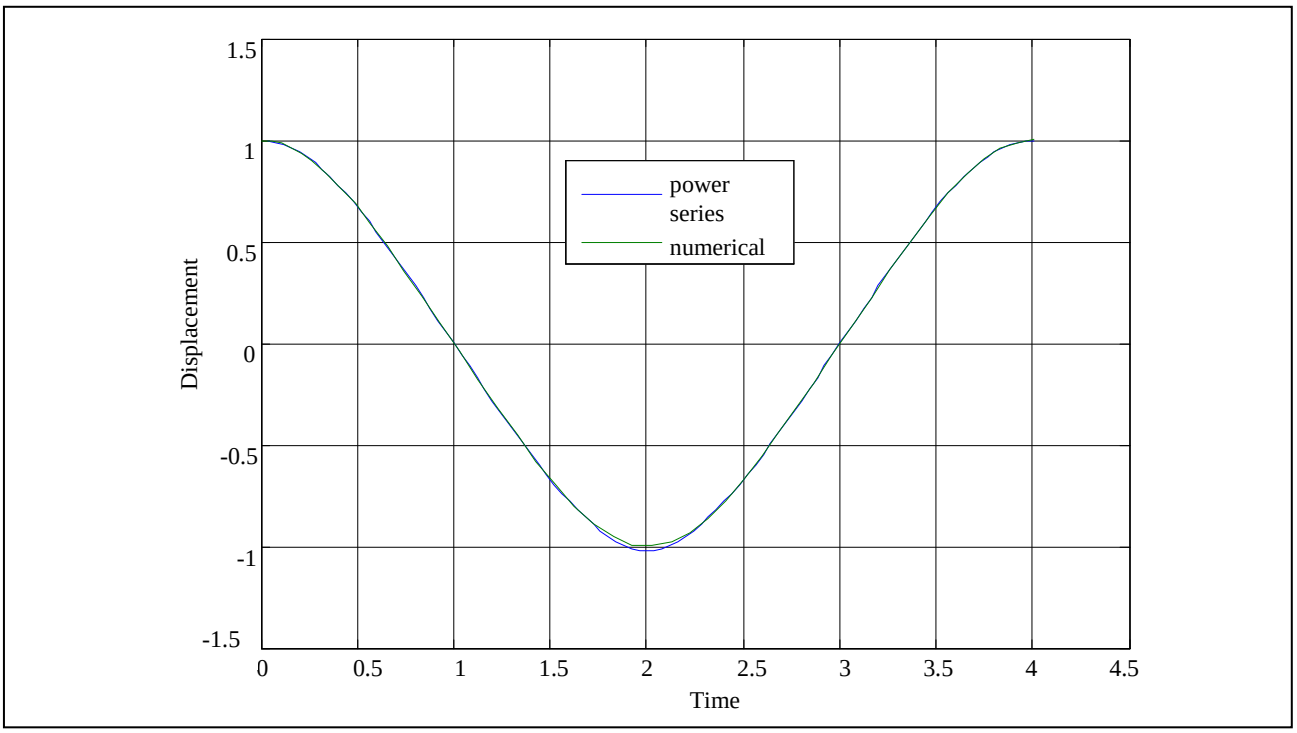

Fig. 2 Displacement response (A = 1, ϵ = 2).

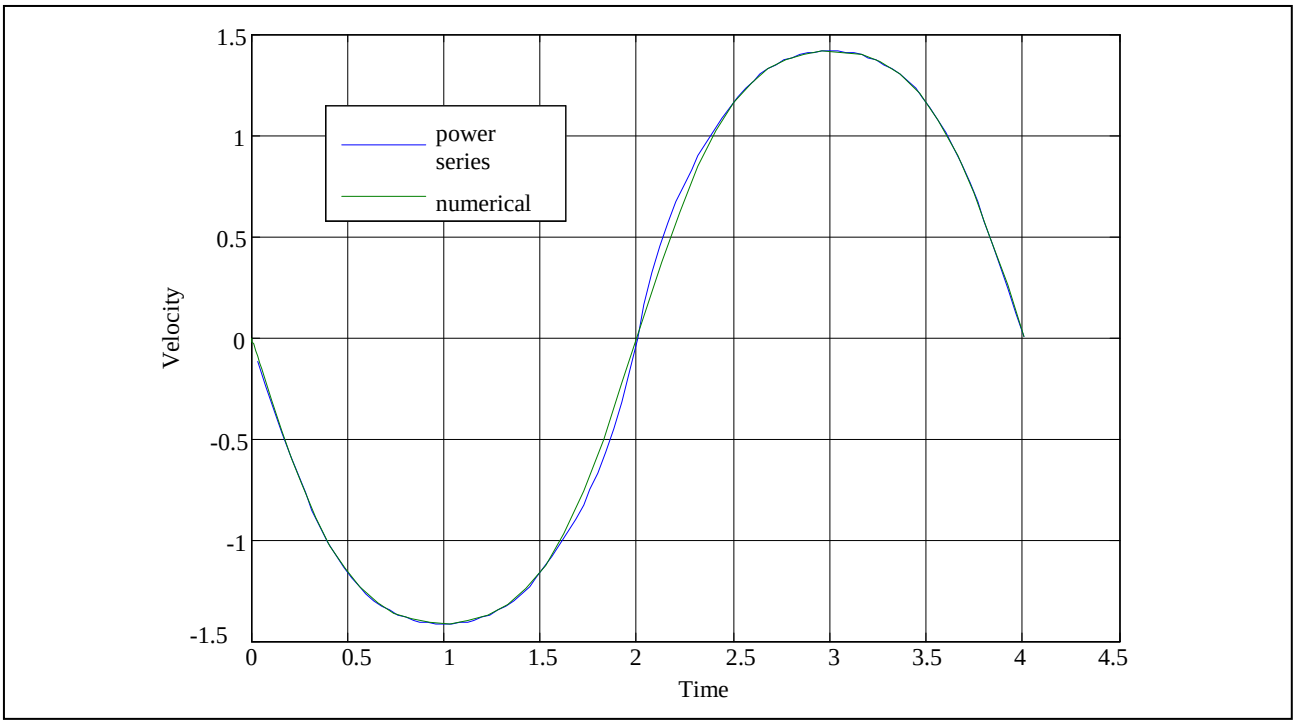

Fig. 3 Velocity response (A = 1, ϵ = 2).

Table 2 Power series coefficients (A = 1, ϵ = 2).

i	a_i	a_{i+5}	a_{i+10}	a_{i+15}	a_{i+20}	a_{i+25}
1	1.000	-1.3047	0.7538	-0.4456	0.2632	-0.1554
2	-2.4367	1.1369	-0.6794	0.4011	-0.2369	0.1399
3	1.4967	-1.0409	0.6110	-0.3610	0.2132	-0.1259
4	-1.6980	0.9285	-0.5502	0.3249	-0.1919	0.1133
5	1.3620	-0.8397	0.4951	-0.2924	0.1727	-0.1020

4. Conclusion

A comparison has been presented between the power series approach and the modified Lindstedt-Poincare method when applied to a strongly nonlinear Duffing oscillator. The results show that the present approach provides extremely accurate vibration frequencies over a wide range of the nonlinear parameter, compared with errors of up to nine percent for the modified L-P method. The time response of the present approach is in excellent agreement with the numerical solution.

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